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Boundary breathers in the sinh–Gordon model

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Abstract. We present an investigation of the boundary breather states of the sinh–Gordon model restricted to a half-line. The classical boundary breathers are presented for a two-parameter family of integrable boundary conditions. Restricting to the case of boundary conditions which preserve the $\phi \rightarrow -\phi$ symmetry of the bulk theory, the energy spectrum of the boundary states is computed in two ways: firstly, by using the bootstrap technique and subsequently, by using a WKB approximation. Requiring that the two descriptions of the spectrum agree with each other allows a determination of the relationship between the boundary parameter, the bulk coupling constant and the parameter appearing in the reflection factor derived by Ghoshal to describe the scattering of the sinh–Gordon particle from the boundary.

1. Introduction

In recent years, there has been renewed interest in field theories defined on restricted domains. In particular, integrable two-dimensional models, for example affine Toda field theories, may be confined to a half-line or an interval by boundary conditions which maintain integrability [3, 5, 16, 25] (for a partial review, see [7]). The variety of possibilities is intriguing, although in most Toda theories the freedom to choose boundary conditions is severely limited to a finite, discrete set of possibilities. In fact, within the models based on the $a_n^{(1)}$, $d_n^{(1)}$ or $e_n^{(1)}$ data, only the model based on $a_1^{(1)}$, the sinh–Gordon model, allows parameters to be introduced as part of the boundary conditions [1].

An outstanding question concerns the quantum integrability of models with boundaries and although there has been some progress towards understanding particular examples, mostly within the class of models based on the a_n series, there remains much to be done to discover the systematics underpinning the apparently bewildering variety of cases.

Even within the sinh/sine–Gordon model, about which so much is now known, there remain some open questions. Up to the present there appears to be a gap in the understanding of how the boundary data, which is prescribed in order to formulate the boundary conditions of the model, is related to the parameters appearing in the family of reflection factors describing particle-boundary scattering. Finding this relationship needs the answers to dynamical questions which cannot be resolved by general requirements such as the reflection Yang–Baxter equation, or ‘crossing-unitarity’. In this paper we shall examine the sinh–Gordon model restricted to a half-line by boundary conditions preserving its bulk symmetry and for

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which one expects boundary bound states, and we shall approach the boundary bound states from two points of view. On the one hand, we will calculate their spectrum using a semiclassical approach rooted in the classic work of Dashen *et al* [12] while, on the other hand, we will compute the same data using bootstrap techniques. The marriage of the two approaches will yield strong evidence for a conjectured relationship between the reflection factors and the boundary data.

2. The sinh–Gordon model on the half-line

The sinh–Gordon model describes a single real scalar field ϕ in $1 + 1$ dimensions with exponential self-interaction. The field equation is

$$\partial_t^2 \phi - \partial_x^2 \phi + \frac{\sqrt{8} m^2}{\beta} \sinh(\sqrt{2} \beta \phi) = 0 \quad (2.1)$$

where m and β are parameters and we have used normalizations customary in affine Toda field theories of which the sinh–Gordon model is the simplest example [2]. The dimensional mass parameter m will be set to unity.

In contrast to the sine–Gordon model with its soliton and breather solutions the sinh–Gordon model has only one real non-singular classical solution, namely the constant vacuum solution $\phi = 0$. In the quantum theory the small oscillations around this vacuum correspond to the sinh–Gordon particle.

The sinh–Gordon model is integrable which implies, in particular, that there are infinitely many independent conserved charges $Q_{\pm s}$, where s is any odd integer, and the S -matrix describing the scattering of n sinh–Gordon particles factorizes into a product of $n(n-1)/2$ two-particle scattering amplitudes. The scattering between two particles of relative rapidity Θ is conjectured to be given by the S -matrix factor [14, 28]

$$S(\Theta) = -\frac{1}{(B)(2-B)} \quad (2.2)$$

where we have used the convenient block notation [2]

$$(x) = \frac{\sinh(\frac{1}{2}\Theta + \frac{1}{4}i\pi x)}{\sinh(\frac{1}{2}\Theta - \frac{1}{4}i\pi x)} \quad (2.3)$$

and the coupling constant B is related to the bare coupling constant β by $B = 2\beta^2/(4\pi + \beta^2)$. Traditionally, scattering and other properties of the sinh–Gordon model have been obtained from knowledge of the lowest breather in the sine–Gordon model by analytic continuation in the coupling constant (but, see also [26]).

The sinh–Gordon model can be restricted to the left half-line $-\infty \leq x \leq 0$ without losing integrability by imposing the boundary condition

$$\partial_x \phi|_0 = \frac{\sqrt{2}m}{\beta} \left(\varepsilon_0 \exp\left[-\frac{\beta}{\sqrt{2}}\phi(0, t)\right] - \varepsilon_1 \exp\left[\frac{\beta}{\sqrt{2}}\phi(0, t)\right] \right) \quad (2.4)$$

where ε_0 and ε_1 are two additional parameters [16, 21]. This set of boundary conditions generally breaks the reflection symmetry $\phi \rightarrow -\phi$ of the sinh–Gordon model. However, the symmetry is preserved when $\varepsilon_0 = \varepsilon_1 \equiv \varepsilon$ and much of this paper will be devoted to this special case.

To describe the sinh–Gordon particles on the half-line one needs in addition to the two-particle scattering amplitude (2.2) also the amplitude for the reflection of a single particle from

the boundary. This reflection amplitude can be deduced from the lowest breather reflection amplitude in the sine–Gordon model by analytic continuation in the coupling constant (i.e. the continuation $\lambda \rightarrow -2/B$ in the notation of [16]). Using the breather reflection amplitudes calculated by Ghoshal [17] gives[†]

$$K(\theta, \varepsilon_0, \varepsilon_1, \beta) = \frac{(1)(2 - B/2)(1 + B/2)}{(1 - E(\varepsilon_0, \varepsilon_1, \beta))(1 + E(\varepsilon_0, \varepsilon_1, \beta))(1 - F(\varepsilon_0, \varepsilon_1, \beta))(1 + F(\varepsilon_0, \varepsilon_1, \beta))} \quad (2.5)$$

where we are again using the block notation from (2.3) but in (2.5) θ represents the rapidity of a single particle. When the bulk reflection symmetry is preserved one of the two parameters E or F vanishes. We shall choose $F = 0$, and consequently one obtains

$$K_0(\theta, \beta) \equiv K(\theta, \varepsilon, \beta) = \frac{(2 - B/2)(1 + B/2)}{(1)(1 - E(\varepsilon, \beta))(1 + E(\varepsilon, \beta))} \equiv K_D \frac{1}{(1 - E)(1 + E)}. \quad (2.6)$$

Actually, the first factor, K_D is the reflection factor corresponding to the Dirichlet boundary condition $\phi(0, t) = 0$, as noted in [16, 17]. All reflection factors satisfy the crossing-unitarity relation which, in the case of scalar reflection factors, reads

$$K\left(\theta + \frac{1}{2}i\pi\right)K\left(\theta - \frac{1}{2}i\pi\right)S(2\theta) = 1. \quad (2.7)$$

The Dirichlet reflection factor K_D satisfies (2.7) by itself.

In this paper we note that contrary to the situation on the whole line, the sinh–Gordon equation restricted to a half-line by integrable boundary conditions has non-singular, finite energy, breather solutions. After quantizing, these will lead to a spectrum of boundary bound states which ought to match not only the physical strip poles of the expression (2.6) but also the poles appearing in similar expressions derived from (2.6) using the boundary bootstrap. These derived reflection factors will be determined below and represent the sinh–Gordon particle reflecting from the excited boundary states. Matching the two ways of looking at the energies of the excited states will determine a relation between ε , β and E , see equation (5.29). In fact the relationship between the two parameters coincides with a tentative suggestion made in [8].

A similar analysis is feasible in the general case ($\varepsilon_0 \neq \varepsilon_1$) but it will not be carried out here. However, the associated boundary breathers and a few of their properties will be described as an essential preliminary to a fuller investigation.

3. Boundary breathers

The sinh–Gordon model on the whole line has no non-singular real solutions other than $\phi = 0$. However, there are singular real breather solutions satisfying the boundary condition (2.4) whose singularities can be designed to lie for all time on the right half-line ($x > 0$). Thus, they are well defined periodic solutions of the sinh–Gordon model on the left half-line and we shall call them *boundary breathers*. Following Hirota, with suitable choices of τ_j , the solutions can be written conveniently in the form [18]

$$\phi = \frac{\sqrt{2}}{\beta} \ln \frac{\tau_0}{\tau_1}. \quad (3.1)$$

[†] In Ghoshal's notation $E = B\eta/\pi$, $F = iB\vartheta/\pi$.

For the symmetrical boundary conditions with $\varepsilon_0 = \varepsilon_1 = \varepsilon$, appropriate choices are

$$\tau_j = 1 + (-1)^j 2 \cos(2t \sin \rho) e^{2x \cos \rho} \frac{1}{\tan \rho} \sqrt{\frac{\varepsilon + \cos \rho}{\varepsilon - \cos \rho}} - e^{4x \cos \rho} \left(\frac{\varepsilon + \cos \rho}{\varepsilon - \cos \rho} \right) \quad (3.2)$$

where the parameter ρ determines the frequency of the breather.

In order for the τ functions to be real, the square root appearing in (3.2) must be, which in turn requires that $|\varepsilon| \geq \cos \rho$. Also, the solution will be singular whenever one of the τ functions is zero. While singularities cannot be avoided entirely it is possible to ensure that there are none in the region $x \leq 0$, and that is sufficient for the present purpose. Noting that for a particular x singularities cannot occur at any time provided

$$1 < \left| \frac{\tan \rho (1 - e^{4 \cos \rho x} ((\varepsilon + \cos \rho)/(\varepsilon - \cos \rho)))}{2e^{2 \cos \rho x} \sqrt{(\varepsilon + \cos \rho)/(\varepsilon - \cos \rho)}} \right| \quad (3.3)$$

we deduce that requiring there are no singularities on the left half-line is equivalent to the restrictions

$$-1 < \varepsilon < 0 \quad \text{and} \quad \cos \rho < -\varepsilon. \quad (3.4)$$

Note that at $\cos \rho = -\varepsilon$ the solution collapses to the vacuum solution $\phi = 0$, indicating that there is a minimum allowed frequency for a breather which is strictly greater than zero. This is a distinctive feature not shared by the usual breathers of the sine-Gordon model on the full line whose frequencies may approach zero.

The energy functional of the sinh-Gordon model with boundary condition (2.4) is

$$\begin{aligned} \mathcal{E}[\phi] = & \int_{-\infty}^0 dx \left(\frac{1}{2} \dot{\phi}^2 + \frac{1}{2} \phi'^2 + \frac{2}{\beta^2} (\cosh(\sqrt{2}\beta\phi) - 1) \right) \\ & + \frac{2}{\beta^2} \left(\varepsilon_0 \left(\exp \left[-\frac{\beta}{\sqrt{2}} \phi(0, t) \right] - 1 \right) + \varepsilon_1 \left(\exp \left[\frac{\beta}{\sqrt{2}} \phi(0, t) \right] - 1 \right) \right) \end{aligned} \quad (3.5)$$

but it is most easily calculated in terms of the τ functions as a boundary term [10],

$$\mathcal{E}[\phi] = \frac{2}{\beta^2} \left(\varepsilon_0 \left(\frac{\tau_0}{\tau_1} - 1 \right) + \varepsilon_1 \left(\frac{\tau_1}{\tau_0} - 1 \right) - \left(\frac{\tau'_0}{\tau_0} + \frac{\tau'_1}{\tau_1} \right) \right) \Big|_{x=0}. \quad (3.6)$$

The energy of the real boundary breather turns out to be given by

$$\mathcal{E}_{\text{breather}} = \frac{8}{\beta^2} (-\cos \rho - \varepsilon) \quad (3.7)$$

and the condition (3.4) ensures that $\mathcal{E}_{\text{breather}}$ is always positive, or zero if $\cos \rho = -\varepsilon$.

In the quantum theory, the continuum of boundary breather solutions is expected to lead to a discrete spectrum of boundary states. To obtain an estimate for the energies of these boundary states one might in the first instance use the Bohr-Sommerfeld quantization condition (see, for example, [4]). One proceeds directly by calculating the left-hand side of

$$\int_0^T dt \int_{-\infty}^0 dx \pi(x, t) \dot{\phi}(x, t) = (2n + 1)\pi \quad (3.8)$$

where $\pi(x, t) = \dot{\phi}$ is the momentum conjugate to ϕ , $T = \pi / \sin \rho$ is the period of the breather and n is an integer. (As usual, we have put $\hbar = 1$.)

It is convenient to set $\varepsilon = \cos \pi a$, $1 > a > \frac{1}{2}$, implying that $\cos^{-1}(-\varepsilon) = \pi(1 - a)$, then integrating gives

$$\int_0^T dt \int_{-\infty}^0 dx \dot{\phi}^2 = \frac{8\pi}{\beta^2} (\rho - \pi(1 - a)). \quad (3.9)$$

The energy levels follow from (3.8) yielding

$$\mathcal{E}_n = \frac{8}{\beta^2} \left(-\cos \pi a + \cos \pi \left(\left(n + \frac{1}{2} \right) \frac{\beta^2}{4\pi} - a \right) \right). \tag{3.10}$$

Since the breathers approach the vacuum solution as $\rho \rightarrow \pi(1 - a)$ by reducing their amplitudes to zero rather than their frequencies, it is natural that the boundary breather spectrum should have a zero point energy. This is the reason for the postulated form of the right-hand side of (3.8).

The difference between two consecutive bound state energies is readily deduced from (3.10) and is conveniently written as

$$\mathcal{E}_{n+1} - \mathcal{E}_n = \frac{16}{\beta^2} \sin\left(\frac{1}{8}\beta^2\right) \cos \frac{\pi}{2} \left(\left(n + 1 \right) \left(\frac{\beta^2}{2\pi} \right) - 2a + 1 \right). \tag{3.11}$$

It will be seen below that this has the form we would have anticipated from the boundary bootstrap. However, given that we know that the coupling constant renormalizes, and we expect the boundary coupling to renormalize too [22], the outcome of this calculation can be at best an indication. A more reliable method for quantizing the boundary breathers is likely to be an adaptation of the techniques developed by Dashen *et al* [12] and this will be pursued in section 5.

Finally, we shall end this section with a brief description of the boundary breathers in those cases where the boundary conditions break the bulk symmetry. As before, the solutions have the general form indicated by (3.1) but this time the two tau functions are more elaborate and are given by

$$\begin{aligned} \tau_j = 1 + (-1)^j & \left(2 \frac{s}{\tan \rho} \cos(2t \sin \rho) \exp(2x \cos \rho) \right. \\ & + \frac{r}{\tan^2(\rho/2)} \exp(2x) - r s^2 \tan^2 \frac{\rho}{2} \exp 2x(2 \cos \rho + 1) \\ & \left. - \left(2 \frac{rs}{\tan \rho} \cos(2t \sin \rho) \exp 2x(\cos \rho + 1) + s^2 \exp(4x \cos \rho) \right) \right) \end{aligned} \tag{3.12}$$

where

$$\begin{aligned} r &= \frac{\sin \frac{1}{2}\pi a_0 - \sin \frac{1}{2}\pi a_1}{\sin \frac{1}{2}\pi a_0 + \sin \frac{1}{2}\pi a_1} \\ s^2 &= \frac{(1 + \cos \rho)(\cos \frac{1}{2}\pi(a_0 + a_1) + \cos \rho)(\cos \frac{1}{2}\pi(a_0 - a_1) - \cos \rho)}{(1 - \cos \rho)(\cos \frac{1}{2}\pi(a_0 + a_1) - \cos \rho)(\cos \frac{1}{2}\pi(a_0 - a_1) + \cos \rho)} \end{aligned} \tag{3.13}$$

and a_0 and a_1 are related to the boundary parameters by

$$\varepsilon_0 = \cos \pi a_0 \quad \varepsilon_1 = \cos \pi a_1.$$

Numerical investigation of these boundary breathers indicates that they are non-singular in the region $x < 0$ provided

$$0 < \cos \rho < -\cos \frac{1}{2}\pi(a_0 + a_1) \quad \cos \frac{1}{2}\pi(a_0 + a_1) < 0 \quad \cos \frac{1}{2}\pi(a_0 - a_1) > 0$$

their energies are given by

$$\mathcal{E} = \frac{4}{\beta^2} \left(-2 - 2 \cos \rho + \left(\sin \frac{1}{2}\pi a_0 + \sin \frac{1}{2}\pi a_1 \right)^2 \right). \tag{3.14}$$

Again, the breathers have frequencies bounded from below because the (a_0, a_1) parameters are restricted. For example, they could lie within the ranges $-1 < a_0 - a_1 < 1$, $1 < a_0 + a_1 < 2$ in the positive quadrant. The boundary breathers for boundary conditions preserving the symmetry of the sinh–Gordon equation are included as the special case $a_0 = a_1$. The possibility $a_0 = -a_1$ is outside the range.

These solutions may be considered as a superposition of a static ‘soliton’ and a ‘boundary breather’, carefully designed to be real and non-singular, and to satisfy the general boundary condition (2.4). They are sinh–Gordon counterparts of the sine–Gordon solutions considered by Saleur *et al* [24].

4. The boundary bootstrap

For certain ranges of values of the parameters E and F the particle reflection amplitude (2.5) has simple poles at particular values of θ on the physical strip, $0 < \text{Im } \theta < \pi/2$. For the case of $F = 0$ these must† be due to the propagation of virtual excited boundary states. The amplitudes for the reflection of the sinh–Gordon particle from these excited boundary states is obtained by the boundary bootstrap [5, 15, 16]. When the reflection factor (2.5) has a pole at $\theta = i\psi$ with $0 < \psi < \pi/2$ then the reflection factor corresponding to the associated excited boundary state is calculated via the relation

$$K_1(\theta) = K_0(\theta)S(\theta - i\psi)S(\theta + i\psi) \quad (4.1)$$

where $S(\theta)$ is the two-particle S -matrix (2.2). Also, since energy is conserved, the energy of the excited boundary state is given by

$$\mathcal{E}_1 = \mathcal{E}_0 + m(\beta) \cos \psi \quad (4.2)$$

where $m(\beta)$ is the mass of the sinh–Gordon particle.

Considering the case $F = 0$, the regions in E where the amplitude (2.6) has poles on the physical strip are

$$\begin{aligned} \text{(I)} \quad & 2 > E > 1 \\ \text{(II)} \quad & -2 < E < -1 \end{aligned} \quad (4.3)$$

since $0 \leq B \leq 2$, the other factors (in K_D) never have poles in the physical strip. In region I, $\psi = \pi(E - 1)/2$ and, using (4.1), we derive the reflection factor for the first excited state,

$$K_1 = K_D \frac{1}{(1 - E)(1 + E)} \frac{(1 + E + B)(1 - E - B)}{(1 - E + B)(1 + E - B)}. \quad (4.4)$$

This in turn has a new pole at $\psi = \pi(E - 1 - B)/2$ provided $B < E - 1$, indicating another excited state whose reflection factor is

$$K_2 = K_D \frac{1}{(1 - E + B)(1 + E - B)} \frac{(1 + E + B)(1 - E - B)}{(1 - E + 2B)(1 + E - 2B)}. \quad (4.5)$$

Continuing in this vein leads to a set of excited states with associated reflection factors given by

$$K_n = K_D \frac{1}{(1 - E + (n - 1)B)(1 + E - (n - 1)B)} \frac{(1 + E + B)(1 - E - B)}{(1 - E + nB)(1 + E - nB)}. \quad (4.6)$$

† Because there is no three-point coupling in the sinh–Gordon model with a symmetrical boundary condition, simple poles on the physical strip can never be due to a generalized Coleman–Thun mechanism [13].

Note that the pole corresponding to the $(n + 1)$ st state will be within the correct range provided E satisfies $2 > E > 1 + nB$. Thus, for a given E and B there can be at most a finite number of bound states, and possibly none. Note too that the reflection factor for scattering from the n th bound state also contains a pole corresponding to the $(n - 1)$ st bound state. We shall see that there is a subtlety concerning the coefficient of this pole because it develops a zero at an n -dependent critical value of β .

The energies of the boundary states are given by repeatedly applying (4.2) and satisfy

$$\mathcal{E}_{n+1} = \mathcal{E}_n + m(\beta) \cos \frac{1}{2}\pi(nB - E + 1). \tag{4.7}$$

This is the result that we wish to compare with the quantization of the classical breather spectrum in order to determine $E(\varepsilon, \beta)$ and $m(\beta)$. However, we shall defer the comparison until after we have developed the Dashen *et al*'s argument in the present context.

5. Semiclassical quantization

To carry out the semiclassical calculation it is first necessary to solve the sinh–Gordon equation linearized in the presence of the boundary breathers. Setting $\phi = \phi_0 + \eta$, the linear wave equations for the fluctuations are

$$\frac{\partial^2 \eta}{\partial t^2} - \frac{\partial^2 \eta}{\partial x^2} + 4\eta \cosh \sqrt{2}\beta\phi_0 = 0 \quad \left(\frac{\partial \eta}{\partial x} + 2\varepsilon\eta \cosh \frac{\beta\phi_0}{\sqrt{2}} \right)_{x=0} = 0. \tag{5.1}$$

It is convenient to solve (5.1) by perturbing (3.1); in other words, we take

$$\eta = \frac{\tau_1 \delta \tau_0 - \tau_0 \delta \tau_1}{\tau_0 \tau_1} \tag{5.2}$$

with $\delta \tau_j$ chosen as follows:

$$\begin{aligned} \tau_j + \delta \tau_j = & 1 + (-)^j (e_1 + e_2 + E_1 + E_2) + A_{12} E_1 E_2 + e_1 (\mu_{11} E_1 + \mu_{12} E_2) \\ & + e_2 (\mu_{21} E_1 + \mu_{22} E_2) + (-)^j A_{12} E_{12} (\mu_{11} \mu_{12} e_1 + \mu_{21} \mu_{22} e_2). \end{aligned} \tag{5.3}$$

In (5.3) we have made use of the Hirota expression for the general multi-soliton solution to the sine–Gordon equation [18, 19], suitably adapted to solve the sinh–Gordon equation, keeping terms up to first order in e_1 and e_2 . The various quantities are given by:

$$\begin{aligned} e_1 = \lambda_1 e^{-i\omega t + ikx} \quad e_2 = \lambda_2 e^{-i\omega t - ikx} \quad \omega^2 - k^2 = 4 \\ E_1 = \exp(2x \cos \rho + 2it \sin \rho + x_0) \quad E_2 = \exp(2x \cos \rho - 2it \sin \rho + x_0) \\ A_{12} = -\tan^2 \rho \quad e^{x_0} = \frac{1}{\tan \rho} \sqrt{\frac{\varepsilon + \cos \rho}{\varepsilon - \cos \rho}} \\ \mu_{11} = \frac{1}{\mu_{22}} = \frac{4 + 2\omega \sin \rho - 2ik \cos \rho}{-4 + 2\omega \sin \rho - 2ik \cos \rho} \quad \mu_{12} = \frac{1}{\mu_{21}} = \frac{-4 + 2\omega \sin \rho + 2ik \cos \rho}{4 + 2\omega \sin \rho + 2ik \cos \rho} \end{aligned} \tag{5.4}$$

where λ_1 and λ_2 are small parameters. Matching the boundary condition at $x = 0$ fixes the ratio λ_2/λ_1 to be

$$K_B = \frac{\lambda_2}{\lambda_1} = \mu_{11} \mu_{12} \frac{ik + 2\varepsilon}{ik - 2\varepsilon} = \frac{(ik + 2 \cos \rho)^2 (ik + 2\varepsilon)}{(ik - 2 \cos \rho)^2 (ik - 2\varepsilon)}. \tag{5.5}$$

In the limit $x \rightarrow -\infty$,

$$\eta \sim \lambda_1 e^{-i\omega t} (e^{ikx} + K_B e^{-ikx}) \tag{5.6}$$

defining the classical reflection factor corresponding to the boundary breather. Taking $\cos \rho = -\varepsilon$, the breather collapses to the vacuum solution $\phi_0 = 0$ and the reflection factor collapses to

$$K_0 = \frac{ik + 2\varepsilon}{ik - 2\varepsilon} \equiv -\frac{1}{(1 - 2a)(1 + 2a)}. \quad (5.7)$$

The ground state reflection factor is easily checked directly and it is the $\beta \rightarrow 0$ limit of the reflection factor given in (2.6). Hence, we may deduce that $E(0) = 2a$.

The classical action of the boundary breather is calculated to be

$$S_{\text{cl}} = \int_0^T dt \int_{-\infty}^0 dx \mathcal{L} = \frac{8\pi}{\beta^2} \left(\rho - \pi(1 - a) + \frac{\cos \rho + \cos \pi a}{\sin \rho} \right) \quad (5.8)$$

and vanishes as it should when $\cos \rho = -\varepsilon$, i.e. $\rho = \pi(1 - a)$.

The period $T = \pi / \sin \rho$ of the boundary breather defines the ‘stability angles’ via

$$\eta(t + T, x) = e^{-iv} \eta(t, x) \equiv e^{-i\omega T} \eta(t, x) \quad (5.9)$$

and the field-theoretical version of the WKB approximation makes use of the stability angles together with a regulator to calculate a quantum action. The standard procedure would be to place the field theory in an interval $[-L, L]$ with periodic boundary conditions and to manipulate the sum over the discrete stability angles thus obtained. However, that option is not available in this case. Instead, it is convenient to treat the sinh–Gordon model in the interval $[-L, 0]$ and to impose the Dirichlet condition $\eta(t, -L) = 0$. Since the limit $L \rightarrow \infty$ will be taken eventually, the stability angles for the boundary breather (v_B), or the vacuum solution (v_0) are effectively determined by the reflection factors given in (5.6) or (5.7), respectively.

Following [12, 23] we need to calculate a sum over the stability angles and use it to correct the classical action. Thus,

$$\Delta = \frac{1}{2} \sum (v_B - v_0) \equiv \frac{1}{2} T \sum \left(\sqrt{k_B^2 + 4} - \sqrt{k_0^2 + 4} \right) \quad (5.10)$$

where k_B and k_0 are the sets of (discrete) solutions to

$$e^{2ik_B L} = -\frac{(ik_B + 2 \cos \rho)^2 (ik_B + 2\varepsilon)}{(ik_B - 2 \cos \rho)^2 (ik_B - 2\varepsilon)} \quad e^{2ik_0 L} = -\frac{ik_0 - 2\varepsilon}{ik_0 + 2\varepsilon}. \quad (5.11)$$

Once Δ is known the quantum action is defined by

$$S_{\text{qu}} = S_{\text{cl}} - \Delta. \quad (5.12)$$

One way to proceed is to note that for large k the solutions to either of (5.11) are close to

$$k_n = \left(n + \frac{1}{2}\right) \frac{\pi}{L}$$

and so it is reasonable to set $(k_B)_n = (k_0)_n + \kappa((k_0)_n)/L$ where, for L large, the function κ is given approximately by

$$e^{2i\kappa(k)} = \frac{(ik + 2 \cos \rho)^2 (ik + 2\varepsilon)^2}{(ik - 2 \cos \rho)^2 (ik - 2\varepsilon)^2}. \quad (5.13)$$

In terms of κ the expression (5.10) is rewritten as

$$\Delta \sim \frac{T}{2L} \sum_{n \geq 0} \frac{(k_0)_n \kappa((k_0)_n)}{\sqrt{(k_0)_n^2 + 4}} + \mathcal{O}(1/L^2)$$

and this, in turn, as $L \rightarrow \infty$ can be converted to a convenient (but actually divergent) integral,

$$\Delta = \frac{T}{2\pi} \int_0^\infty dk \frac{k\kappa(k)}{\sqrt{k^2 + 4}} \tag{5.14}$$

with which we shall have to deal. Note that κ vanishes when $\cos \rho = -\varepsilon$.

Integrating (5.14) by parts we find

$$\Delta = \frac{T}{2\pi} \left(\kappa\sqrt{k^2 + 4} \Big|_0^\infty - \int_0^\infty dk \frac{d\kappa}{dk} \sqrt{k^2 + 4} \right) \tag{5.15}$$

where

$$\frac{d\kappa}{dk} = \frac{4 \cos \rho}{k^2 + 4 \cos^2 \rho} + \frac{4 \cos \pi a}{k^2 + 4 \cos^2 \pi a} \tag{5.16}$$

and we note that with a suitable choice of branch

$$\kappa \sim -\frac{4 \cos \rho}{k} - \frac{4 \cos \pi a}{k} \quad \text{as } k \rightarrow \infty. \tag{5.17}$$

From (5.17) and recalling that $\cos \rho < -\varepsilon$, we deduce that κ approaches zero from above as $k \rightarrow \infty$. Also, from (5.16) it is clear that the derivative of κ is positive near $k = 0$ but negative as $k \rightarrow \infty$. Hence, the first term in (5.15) is well defined and the appropriate branch of κ has $\kappa(0) = 0$. On the other hand, the derivative of κ is not decaying sufficiently rapidly to ensure that the second term in (5.15) is finite. However, this was to be expected since a perturbative analysis of the sinh–Gordon model confined to a half-line needs mass and boundary counter terms to remove logarithmic divergences (which would be removed automatically by normal-ordering the products of fields in the bulk theory). With this in mind, the integral remaining in (5.15) should be replaced by

$$\int_0^\infty dk \sqrt{k^2 + 4} \left(\frac{4 \cos \rho}{k^2 + 4 \cos^2 \rho} - \frac{4 \cos \rho}{k^2 + 4} + \frac{4 \cos \pi a}{k^2 + 4 \cos^2 \pi a} - \frac{4 \cos \pi a}{k^2 + 4} \right) \tag{5.18}$$

the first counter-term removing the bulk divergence and the second being there to remove a similar divergence associated with the boundary. In effect, we are regarding the parameter a as describing the bare coupling which appears in the boundary part of the Lagrangian once it is written in terms of normal-ordered products of fields. The counter-terms clearly respect the symmetry and the whole expression vanishes when $\rho = \pi(1 - a)$. The integrals in (5.18) need to be treated carefully with an eye to the fact that $\cos \rho > 0$ but $\cos \pi a < 0$.

Assembling the various components leads to

$$\Delta = -\frac{2}{\sin \rho} (\cos \rho + \cos \pi a + \rho \sin \rho - \pi(1 - a) \sin \pi a). \tag{5.19}$$

Recalling (5.8), and using (5.19), the quantum action defined in (5.12) is given by an expression of the form

$$S_{\text{qu}} = \frac{4}{B} \left(\frac{\cos \rho}{\sin \rho} + \rho - \frac{\pi}{2} \right) + \frac{8\pi}{\beta^2} \left(\pi a - \frac{\pi}{2} \right) + \frac{\Gamma(a)}{\sin \rho} + \pi \tag{5.20}$$

where Γ is independent of ρ ,

$$\Gamma = \frac{4}{B} \cos \pi a + 2\pi(a - 1) \sin \pi a. \tag{5.21}$$

Once the quantum action is determined the quantum energy is defined by

$$\mathcal{E}_{\text{qu}} = -\frac{\partial S_{\text{qu}}}{\partial T} = \frac{\sin^2 \rho}{\pi \cos \rho} \frac{\partial S_{\text{qu}}}{\partial \rho} = -\frac{4}{\pi B} \cos \rho - \frac{\Gamma}{\pi} \tag{5.22}$$

and the WKB quantization condition states that

$$W_{\text{qu}} = S_{\text{qu}} + T\mathcal{E}_{\text{qu}} = \frac{4}{B}\left(\rho - \frac{1}{2}\pi\right) + \frac{8\pi}{\beta^2}\left(\pi a - \frac{1}{2}\pi\right) + \pi = 2N\pi. \quad (5.23)$$

Here, $N = n + N_0$ with n a positive integer or zero, and we expect N_0 should be $\frac{1}{2}$. Hence, the energies of the quantized boundary breather states are determined by a set of special angles ρ_n ,

$$\rho_n = \frac{\pi}{2}\left(1 + B\left(N - \frac{1}{2}\right) - \frac{2\pi B}{\beta^2}(2a - 1)\right) \quad (5.24)$$

and given by

$$\mathcal{E}_n = -\frac{4}{\pi B}\cos\rho_n - \frac{\Gamma}{\pi} = -\frac{4}{\pi B}\cos\frac{\pi}{2}\left(\left(N - \frac{1}{2}\right)B + 1 - \frac{2\pi B}{\beta^2}(2a - 1)\right) - \frac{\Gamma}{\pi}. \quad (5.25)$$

Notice that as $\beta \rightarrow 0$, $\rho_n \rightarrow \pi(1 - a)$ independently of N . Thus, the frequencies collapse to the lowest allowed frequency, namely $\omega_0 = 2\sin a\pi$. On the other hand, in the same limit the energies are independent of β and non-zero,

$$\mathcal{E}_n \rightarrow N\omega_0. \quad (5.26)$$

This is precisely the spectrum of a harmonic oscillator vibrating at the fundamental frequency ω_0 provided we set $N = n + \frac{1}{2}$. With this interpretation, the vacuum has a non-zero zero-point energy due to the presence of the boundary.

Using (5.25) the corresponding differences in the energy levels are given by

$$\mathcal{E}_{n+1} - \mathcal{E}_n = \frac{8}{\pi B}\sin\frac{\pi B}{4}\cos\frac{\pi}{2}\left(\frac{2\pi B}{\beta^2}(2a - 1) - NB\right). \quad (5.27)$$

Comparing (5.27) with the outcome of the bootstrap calculation (4.7) ought to assist us in identifying the unknown parameter E which appeared in the expression for the reflection factor (2.6). Thus, from the first excited level we deduce

$$E - 1 = \frac{2\pi B}{\beta^2}(2a - 1) - N_0B \equiv (2a - 1)\left(1 - \frac{1}{2}B\right) - N_0B. \quad (5.28)$$

Rearranging, we have

$$E(\varepsilon, \beta) = 2a\left(1 - \frac{1}{2}B\right) + (1 - 2N_0)\frac{1}{2}B. \quad (5.29)$$

Taking the limit as $a \rightarrow \frac{1}{2}$ from above, equation (5.29) is in agreement with the expression given by Ghoshal and Zamolodchikov for the Neumann condition provided $N_0 = \frac{1}{2}$ [16]. With a arbitrary, equation (5.29) agrees both with perturbative calculations to order β^2 given in [8, 27], and with a conjectured all-orders guess reported in [8]. Once N_0 is chosen, the other excited states match up in the two calculations without any further adjustments.

In the bulk sine–Gordon theory the analogous quantity to N_0 vanishes in Dashen *et al*'s calculation of the breather spectrum. In the half-line theory, we have found that the two ways of regarding the spectrum of boundary bound states match provided $N_0 = \frac{1}{2}$. Although we do not yet have an independent reason for expecting this value of N_0 on the basis of WKB theory, its appearance in (5.23) is reminiscent of the extra $\frac{1}{2}$ correction to the Bohr–Sommerfeld quantization condition and it also provides a natural interpretation of the limiting spectrum (5.26)

The comparison with (4.7) also permits us to deduce an expression for $m(\beta)$, the mass of the sinh–Gordon particle:

$$m(\beta) = \frac{8}{\pi B} \sin \frac{\pi B}{4}. \quad (5.30)$$

This is independently interesting. Previously, the same expression for the mass has been deduced via analytic continuation using knowledge of the sine–Gordon breather spectrum on the whole line. However, here it arrives naturally within the context of the sinh–Gordon model itself. It appears that once the model is defined over a restricted region by boundary conditions which permit the existence of boundary states, boundary effects allow bulk parameters to be determined. Notice that periodic boundary conditions, which are in some respects the most natural to impose, and are certainly the traditional choice, do not share this property.

6. Discussion

In this section we need to take another look at the two descriptions of the boundary bound state spectrum. Using what we have learned, the boundary states are described by two different sets of angles which are linear functions of B . From the WKB calculation we have the set ρ_n given by

$$\rho_n = \pi(1 - a) + \frac{1}{2}\pi(n + a - \frac{1}{2})B \quad n = 0, 1, 2, 3, \dots \quad (6.1)$$

The ground state corresponds to ρ_0 and lies in the spectrum for all values of B . This is clear because as B traverses its range from 0 to 2, ρ_0 increases from $\pi(1 - a)$ to $\pi/2$. On the other hand, ρ_n , $n \geq 1$ corresponds to an excited state which will leave the spectrum at some critical value of B when ρ_n attains $\pi/2$. Specifically, the critical couplings are given by

$$B_n^c = \frac{2(2a - 1)}{2n + 2a - 1} \quad \text{or} \quad \frac{\beta_n^{c2}}{4\pi} = \frac{2a - 1}{2n}. \quad (6.2)$$

The other description is derived from the bootstrap. Taking the conjectured form of E , equation (5.29) with $N_0 = \frac{1}{2}$, leads to another set of angles ψ_n defined by

$$\psi_n = \frac{1}{2}\pi(2a - 1 - (a + n - 1)B) \quad n = 1, 2, 3, \dots \quad (6.3)$$

Although these describe the same set of states via the bootstrap, the angles are clearly very different. One striking difference concerns the critical value of the coupling $B_n^{c'}$ at which the state exits the spectrum. The angles (6.3) clearly decrease with increasing B and the critical point is reached when an angle vanishes. Thus, we have

$$B_n^{c'} = \frac{2a - 1}{a + n - 1} \quad \text{or} \quad \frac{\beta_n^{c'2}}{4\pi} = \frac{2a - 1}{2n - 1}. \quad (6.4)$$

The two critical points (6.2) and (6.4) are similar but not the same. Curiously, in terms of the inverse coupling the difference is independent of n :

$$\frac{4\pi}{\beta_n^{c2}} - \frac{4\pi}{\beta_n^{c'2}} = \frac{1}{2a - 1}.$$

The fact that the two critical points are different needs explanation. Unfortunately, we do not have a complete dynamical explanation of this. The problem is that a bound state appears to leave the spectrum before the pole marking it in a reflection factor moves out of range.

Consider the bound state with label n . At the associated rapidity $i\psi_n$ there is a pole in the two reflection factors K_{n-1} and K_n . In the first of these, the pole indicates the possibility

of exciting the state $n - 1$ to the state n ; in the second it indicates the possibility of dropping from state n down to state $n - 1$. In both cases, of course, the process is virtual, but in the second case the process corresponds to a ‘crossed’ diagram. Of the various parts in (4.6), the one which produces the cross-channel pole is $(1 - E + (n - 1)B)$, one of the factors in the denominator. At the critical coupling this is cancelled by the factor $(1 - E + nB)$ because, at the critical value B_n^c , $2E - (2n - 1)B = 2$. This is consistent with a zero in the S -matrix (2.2) at $i\pi B_n^c/2$ which contributes to the cross-channel diagram. For values of the coupling between the two critical values, the pole at $i\psi_n$ in K_{n-1} needs explanation.

That the pole indicating a bound state can persist beyond the value of the coupling at which the bound state ceases to exist is a phenomenon familiar in the breather spectrum of the bulk sine–Gordon model. In the notation we have been using, we simply make the change $\beta \rightarrow i\beta$ and redefine $B(i\beta) = -b$. Then, the n th breather leaves the spectrum at $b = 2/n$. This is typically signalled by the appearance of double poles in S -matrices, rather than the pole position moving across the boundary of the physical strip. However, in this case, the explanation for the pole beyond the critical coupling lies in a Coleman–Thun mechanism using solitons†.

A second point we wish to discuss is the following. Given the expression for E , equation (5.29) with $N_0 = \frac{1}{2}$, we see immediately that if the parameter a is held fixed as $B \rightarrow 2$, then $E \rightarrow 0$, and every reflection factor (4.6) has the same limit

$$K_n \rightarrow -\frac{1}{(1)^2}. \quad (6.5)$$

The latter is the classical reflection factor corresponding to the boundary condition (2.4) with $\varepsilon_0 = \varepsilon_1 = 1$ [6]. It is natural to suppose that the same expression for $E(\varepsilon, \beta)$ will be appropriate for $\varepsilon > 0$ although we cannot prove it. However, if it is the case, then almost all reflection factors will have the property (6.5). The exception to this is the symmetrical Dirichlet condition whose reflection factor has the property $K_D(4\pi/\beta) = K_D(\beta)$; that is, K_D is self-dual. Apart from noting the phenomenon we can offer no explanation as to why one particular nonlinear boundary condition should be singled out to be the limit point of almost all the others. It will be interesting to discover whether this remains so after the complete analysis of the general case $\varepsilon_0 \neq \varepsilon_1$. It is perhaps worth remarking that this special boundary condition is one of the two singled out in the supersymmetric version of the model [20] (for the other $\varepsilon_0 = \varepsilon_1 = -1$). If the expression for E is also correct for $\varepsilon > 0$ then $E(1, \beta) \equiv 0$, indicating that this specially symmetrical boundary condition also has a self-dual reflection factor. Perhaps this is also true for the model with supersymmetry.

In this paper, we have obtained the expression (5.29) for E in terms of the parameter a . However, there is an indication from work on higher $a_n^{(1)}$ Toda theories that the renormalized boundary parameter is not E itself but $G = E + B/2$. In these theories, for $n > 1$, there is only a discrete set of integrable boundary conditions and for many of these the reflection factors are known [11]. These reflection factors can be specialized to the case $n = 1$ which corresponds to the sinh–Gordon model and one obtains the reflection factor (2.6) at fixed (coupling-constant-independent) values of $G = E + B/2$ rather than fixed values of E . Further motivation for regarding $G = E + B/2$ as the physical boundary parameter comes from the study of solitons in the sine–Gordon model on the half-line [9].

Finally, it must be said that the WKB method gives an all-orders result in terms of β and is exact for the bulk sine–Gordon model. Again, we would probably be surprised if that were not the case in the present setting.

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